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Quasiparticles in the Mixed Phase of Superconducting Cuprates: A Semiclassical Green's Function Approach

T. V. Ramakrishnan[1] **and A. K. Rajagopal**[2]

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We consider ($d_{x^2-y^2}$) superconducting state quasiparticles coupled to vortices. Since the perturbation due to the latter varies slowly with distance outside the vortex core, a semiclassical approximation for the quasiparticle Green's function suffices. This is used to discuss several questions of interest, viz. choices of gauge, the quasiparticle density of states in presence of vortices, and finally longitudinal as well as Hall magneto thermal conductivity.

KEY WORDS: High temperature superconductivity; mixed phase; vortices; nodal quasiparticles.

1. INTRODUCTION

Fourteen years after the discovery of superconductivity in cuprates, it is increasingly clear that their normal ($T > T_c$) state is unique among solids.[(1)] It is metallic in plane and insulating perpendicular to it, has poorly defined but mobile electronic excitations in plane, and has universal power law temperature dependence for transport quantities, without a quantum scale. These and many other characteristic properties are widely believed to reflect the non Fermi liquid (Luttinger liquid?) nature of these doped strongly correlated Mott insulators[(1)] though a detailed theory does not exist. By contrast, the superconducting state seems in effect to be BCS like, with a $d_{x^2-y^2}$ symmetry gap function that has nodes and sign changes on the Fermi surface, and with well defined quasiparticle excitations. How such a superconducting phase arises out of the normal state is not clear. A natural question is whether all the observed electronic properties in the superconducting state can be described in terms of the BCS condensate and

[1] Centre for Condensed Matter Theory, Department of Physics, Indian Institute of Science, Bangalore 560 012, India.
[2] Naval Research Laboratory, Washington DC 20375-5000.

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nodal quasiparticle excitations, or whether effects of the strange normal state show up in some phenomena below T_c, and if so how. The first question is nontrivial because the existence of point nodes in the gap Δ_k implies highly anisotropic zero gap quasiparticle excitations ("Dirac" quasiparticles) which are strongly affected by perturbations such as magnetic fields, impurities, and temperature, in ways often qualitatively different from the more familiar s-wave BCS superconductor. These ways are not fully understood yet, as is evident from several unusual equilibrium and transport phenomena in the superconducting phase which continue to be puzzling.

Against this background, we explore here the dynamics of a $d_{x^2-y^2}$ symmetry superconductor in a magnetic field H ($\ll H_{c_2} \simeq 150T$), which enters the system as supercurrent vortices or quantized magnetic flux tubes. The Green's functions are obtained using an equation of motion method, and a number of consequences are discussed. In order to motivate the discussion, we first describe some effects predicted, as well as observations of equilibrium and transport phenomena in the mixed phase (Section 2). We then obtain the equations satisfied by the Green's function (Section 3), and use them to discuss several questions of current interest. First, we look into the question of whether in the presence of a (relatively small) magnetic field, extended plane wave like states continue to be a good basis for describing the effect of the magnetic field, or whether the eigenstates are localized Landau like levels.(2) The two extremely different limiting possibilities mentioned above arise formally from different ways of including the effect of the 2π phase rotation (per pair) caused by a vortex. Instead of analyzing the effective quasiparticle Hamiltonian in different singular gauges, we investigate the Green's function. We show, after appropriate transformations preserving single valuedness, that Green's functions in different gauges lead to the same equation of motion, one most simply described in terms of plane wave quasiparticle states, at least for weak fields.

In the next section (Section 4), we obtain the quasiparticle density of states from the Green's function. In the semiclassical approximation used, it is exactly the same as the Doppler shifted density proposed by Volovik(3) using a Bogoliubov–de Gennes quasiparticle approach. We propose a systematic cumulant expansion method for evaluating the density of states for an arbitrary dense arrangement of vortices interacting with quasiparticles. A number of consequences are pointed out,(4) including the possibility of a novel electronically driven vortex fluid-solid transition. Finally (Section 5), we discuss the Boltzmann transport equation for quasiparticles in the presence of vortices, and (say) a thermal gradient. The supercurrent due to vortices changes slowly with distance (in comparison to the quasiparticle de Broglie wavelength) so that the semiclassical Kadanoff–Baym approach(5) used here is appropriate. A semiclassical transport equation for an s-wave superconductor has been derived and used by several authors; see Aronov *et al.*[6] for a review.

In most physical systems, quasiparticle relaxation originates from relatively short range scattering introduced into the equation for the Green's function separately, besides a static potential which varies smoothly with position and in which the quasiparticle is locally equilibrated. Here, the effective vector potential due to the supercurrent varies slowly in space; it affects the quasiparticle spectral density, i.e., the local quasiparticle energy, and *also* causes small angle scattering of quasiparticles, determining the relaxation of energy current carried by them. We do not solve the latter semiclassical scattering problem here. In the Born approximation for scattering, quasiparticle transport has been recently analyzed by Mandal and Ramakrishnan[7] to explain the unusual magnetic field and temperature dependence of the thermal conductivity. We point out here that the observed unusual Hall or transverse thermal conductivity behavior in the cuprate superconductors (see below) has its origin in the spatially fluctuating Lorentz force; this term is present in the semiclassical transport equation.

2. QUASIPARTICLES IN THE MIXED PHASE OF A $d_{x^2-y^2}$ SUPERCONDUCTOR: THEORY AND EXPERIMENT

We briefly summarize here some theoretical predictions and experimental observations connected with electronic properties of cuprate superconductors in the mixed phase. The Green's function formalism and its applications presented later below are motivated by these results.

The gap function in the superconducting state of the cuprates is well approximated by[8]

$$\Delta_k = \Delta_o[\cos(k_x a) - \cos(k_y a)] \tag{1a}$$

where (k_x, k_y) are the two dimensional Fermi surface coordinates. For the square Cu lattice in cuprates, the real space form of the gap function Eq. (1a) is obviously

$$\Delta_{\vec{i},\vec{j}} = \frac{\Delta_o}{2}\,[\delta_{\vec{j},\vec{i}\pm\hat{e}_x a} - \delta_{\vec{j},\vec{i}\pm\hat{e}_y a}] \tag{1b}$$

namely nearest neighbor pairs with a sign difference between x and y pair bonds. The quasiparticle excitation energy is

$$E^o_k = \pm\sqrt{\tilde{\varepsilon}^2_k + \Delta^2_k} \tag{2}$$

as in a BCS superconductor. The Fermi surface is defined by $\tilde{\varepsilon}_k = 0$, and has a hole like shape, as seen in ARPES (angle resolved photoemission spectroscopy) experiments.[9] Near the nodes of Δ_k, namely near $k_x = \pm k_y$, and the zeroes of $\tilde{\varepsilon}_k$ (i.e., the Fermi surface), E_k^o can be expanded to leading order in the deviation from zero, and one has

$$E_k^o = \pm\sqrt{v_F^2 k_1^2 + v_\Delta^2 k_2^2} \tag{3}$$

where $\tilde{\varepsilon}_k = v_F k_1$ and $\Delta_k = v_\Delta((k_x \pm k_y)/\sqrt{2}) = v_\Delta k_2$. The density of quasiparticle states per unit area for low energy E is seen from Eq. (3) to be

$$\rho(E) = (\pi v_F v_\Delta)^{-1}\, E \tag{4a}$$

In the presence of vortices and the associated superflow, the kinetic momentum of an electron in the superconductor is $(\vec{p} + (m\vec{v}_s(\vec{r})/2))$ rather than $\vec{p}$, where

$$m\vec{v}_s(\vec{r}) = \{\hbar\, \vec{\nabla}\tilde{\theta} - (2e\vec{A}/c)\} \tag{4b}$$

(for a well defined quasiparticle with a free electron like kinetic energy). Here $\tilde{\theta}$ is the phase of a pair at cm coordinate $\vec{r}$, due to vortices. The consequent change in quasiparticle energy is seen to be (Volovik[3])

$$E_{\vec{p}} = E_{\vec{p}}^o + \vec{p} \cdot \vec{v}_s(\vec{r}) \tag{5}$$

to first order in $\vec{v}_s(\vec{r})$ and on neglecting the noncommutativity of $\vec{p}$ and $\vec{v}_s(\vec{r})$. The quasiparticle density of states is then

$$\rho(v) = \sum_p \langle \delta(v - E_p^o - \vec{p} \cdot \vec{v}_s(\vec{r})) \rangle = \sum_p \rho_p(v) \tag{6}$$

In Eq. (6), the symbol denotes averaging over vortex locations. Volovik pointed out that because the characteristic magnitude of $m\vec{v}_s(\vec{r})$ is $\sim \hbar\, \nabla\tilde{\theta}(\vec{r}) \sim \hbar(\nabla\tilde{\theta}(\vec{r}))_{\vec{r}=\vec{r}^*} \simeq (2\pi\hbar/r^*)$ where r^* is the inter-vortex separation, the density of states inferred from Eq. (4a) and Eq. (6) is

$$\rho(E) \simeq (\pi v_F v_\Delta)^{-1}\, (p_F/mr*) \tag{7}$$

Thus there is a nonzero quasiparticle density of states at zero energy, proportional to $\sqrt{H}$. This implies a specific heat linear in T, but with a coefficient proportional to $(\sqrt{H/H_{c_2}})$ (in contrast to the $H = 0$ electronic specific heat $c_v(T) = aT^2$). Such an unusual linear, $\sqrt{H}$ dependent specific heat has been observed in careful experiments on 123 and other cuprate superconductors.[10] We shall see later below that Eq. (6) is valid under fairly general conditions which we discuss. We also describe there a cumulant expansion method for explicitly obtaining $\rho_p(\nu)$ for an arbitrary distribution of vortices.

However, there is the question of whether the true low energy behavior of the density of states is captured by the semiclassical approximation. Two fully quantum approaches[2; 14, 15] lead to results differing from each other, and from the Volovik limit. The approaches differ in the way the phase rotation due to the order parameter is handled by singular gauge transformation. In the presence of vortices, the nonlocal pair potential becomes

$$\Delta(\vec{r}_1, \vec{r}_2) = \tilde{\Delta}(\vec{r}_1, \vec{r}_2) \exp\left[i \sum_{\ell} \theta\left(\frac{\vec{r}_1 + \vec{r}_2}{2} - \vec{R}_\ell \right) \right] = \tilde{\Delta}(\vec{r}_1, \vec{r}_2) \exp(i\tilde{\theta}) \tag{8}$$

where the $\tilde{\Delta}$ is real, and θ is the polar angle around the vortex centred at $\vec{R}_\ell$. Writing the Bogoliubov–de Gennes equations symbolically as

$$Tu + \Delta v = Eu \tag{9a}$$

$$\Delta^* v - Tu = Ev \tag{9b}$$

where T is the kinetic energy operator and E is the energy eigenvalue, a commonly used gauge transformation[11–13, 6] is

$$\begin{aligned} u &\Rightarrow \tilde{u} \exp(+i\tilde{\theta}/2) \\ v &\Rightarrow \tilde{v} \exp(-i\tilde{\theta}/2) \end{aligned} \tag{10}$$

The Bogoliubov–de Gennes equations for $(\tilde{u}, \tilde{v})$ are solved. In the equations for $\tilde{u}$ and $\tilde{v}$, the effective vector potential, $\vec{A}_{\text{eff}} = (\vec{A} - (ch/2e)\,\vec{\nabla}\tilde{\theta})$ coupled to quasiparticles produces a zero net magnetic field; this is just an expression of perfect London screening.

We notice that the kinetic energy term now becomes

$$T = \frac{1}{2m}\,[\vec{p} + (m\vec{v}_s/2)]^2 \tag{11}$$

where $m\vec{v}_s(\vec{r}) = -(2e\vec{A}_{\text{eff}}/c) = \{\hbar\,\vec{\nabla}\tilde{\theta} - (2e\vec{A}/c)\}$ as defined in Eq. (4b). If used naively, i.e., with a $\tilde{u}$ that does not change sign on going round a vortex an odd number of times, u changes sign, i.e., is double valued. Single valued u's are obtained with $\tilde{u}$ chosen to be *double valued*[11–13] as has been the practice in detailed one vortex calculations. This complication can be avoided in several ways. Gor'kov and Schrieffer[2] effectively and Anderson[2] explicitly choose a gauge in which

$$u \to u^{\text{GSA}} \tag{12a}$$

and

$$v \rightarrow v^{\mathrm{GSA}} \exp(-i\tilde{\theta}) \tag{12b}$$

rather than Eq. (10). So that if u^{GSA} and v^{GSA} are chosen single valued so are u and v. However, with this choice, the equations satisfied by u^{GSA} and v^{GSA} describe a quasiparticle in a net magnetic field:

$$\left[\frac{1}{2m}\left(\vec{p}-\frac{e\vec{A}}{c}\right)^2-\mu\right] u^{\mathrm{GSA}}+\tilde{\Delta} v^{\mathrm{GSA}}=E u^{\mathrm{GSA}} \tag{13a}$$

$$\left[-\frac{1}{2m}(\vec{p}+m\vec{v}_s)^2+\mu\right] v^{\mathrm{GSA}}+\tilde{\Delta} u^{\mathrm{GSA}}=E v^{\mathrm{GSA}} \tag{13b}$$

This choice of singular gauge corresponds to associating a full flux quantum with say the down spin electron, and none with the up spin electron (in contrast to the transformation Eq. (10), which implies half a flux quantum each associated with up and down spin electrons). If the superfluid velocity term mv_s in Eq. (13b) is neglected, the eigenvalues E for a $d_{x^2-y^2}$ superconductor, i.e., with $\Delta_k \simeq (\Delta_o a^2/2)(k_y^2-k_x^2)$ (see Eq. (1a)) are $E_n=\sqrt{\hbar\omega_c\Delta_o}\sqrt{n}$ where $\omega_c=(eH/mc)$ is the cyclotron frequency. Since the neglected $\vec{p}\cdot\vec{v}_s$ perturbation is large and lifts Landau level degeneracy in addition to mixing different Landau levels, it is not clear that the Landau level representation is a good starting point for discussing quasiparticle density of states and quasiparticle transport.

Franz and Tesanovic[14] proposed recently yet another gauge transformation. They divide the vortices into two sublattices A and B (or two equal sized, interspersed groups in general). The phase of group A is associated with say u or up spin particles, and the phase of group B with v or down spin, i.e.,

$$u=u^{\mathrm{FT}} \exp(i\tilde{\theta}_A) \tag{14a}$$

$$v=v^{\mathrm{FT}} \exp(-i\tilde{\theta}_B) \tag{14b}$$

where

$$\tilde{\theta}=(\tilde{\theta}_A+\tilde{\theta}_B)=\sum_{\ell\in A}\theta(\vec{r}-\vec{R}_\ell)+\sum_{j\in B}\theta(\vec{r}-\vec{R}_j) \tag{14c}$$

This ensures the single valuedness of u and v, if u^{FT} and v^{FT} are single valued. It emphasizes the fact that the magnetic field is fully screened, in effect on a length scale of order the intervortex separation, when one has a many vortex system. However, the physical vortex system is somewhat arbitrarily divided into two sublattices. In this gauge, it is again clear that plane wave like quasiparticles move in zero net magnetic field and the spectrum (calculated numerically for periodic vortex arrangements in refs. 14 and 15) has no Landau level like structure.

In our work, we focus on Green's functions and physical quantities calculable from it, and examine questions of single valuedness, gauge transformation etc. for the Green's functions. We show that transformations maintaining single valuedness in centre of mass coordinate relate the Green's function and equations of motion in different gauges, and conclude that the Green's function corresponding to the conventional gauge Eq. (10) has a relatively simple equation of motion, and is single valued. This is used for obtaining gauge invariant density of states and transport equation subsequently.

Experimentally, both equilibrium and transport properties of *d*-wave superconductors in a magnetic field show unusual behavior, in many cases not understood. The linear in T electronic specific heat, with a coefficient proportional to $\sqrt{H}$ (at low T) has already been mentioned.[10] A related $\sqrt{H}$ decrease of the superfluid density or stiffness has been inferred from measurements of the imaginary part of the ac conductivity,[16] and of the penetration depth via μsr.[17]

There are many high quality experimental results (from STM measurements) of position dependent local quasiparticle density of states in the mixed phase. There are no theoretical results that describe the observations fully. Perhaps the most striking experimental result is the existence of one nearly isotropic bound state localized inside the core, with a binding energy of 7–9 meV seen in both $YBa_2Cu_3O_{7-\delta}$[18] and Bi-2212.[19] Davis *et al.*[19] in a beautiful series of experiments, show further that the bound state wave function falls off exponentially with a localization length $\xi_{\rm loc} \simeq 22$ Å. Naively, one expects that in a *d*-wave superconductor, the pair potential well centred around a vortex core has zero depth in four directions so that no truly bound state is possible. Most microscopic calculations do not find a bound state. The common occurrence of the vortex bound state in different systems suggests that the simple picture of a BCS like $d_{x^2-y^2}$ superconductor with a conventional vortex core breaks down due to the strong perturbation caused by the highly inhomogeneous normal state in the small ($\xi \sim 15$ Å) core region.

Perhaps the phenomena hardest to make sense of are transport properties in the mixed phase. Very early, the electrical Hall effect was noted to have a sign anomaly. As pointed out by Nagaoka *et al.*[20] the flux flow Hall effect has an electron like sign in the underdoped and optimally doped regime while the Hall effect is hole like in normal state. Conventional theories for flux flow Hall effect lead to the *same* sign for it as in the normal state. Various attempts to explain this in terms of quasiparticle and vortex dynamics[20] are largely unsuccessful.

Thermal transport in the mixed phase has the advantage that the vortices do not have a net drift (which they do in an electric field) so that an understanding of vortex dynamics is not necessary. The electronic energy (heat) current is carried by quasiparticles (the condensate, being a single coherent state, has no entropy) and relaxes by collisions with impurities, other quasiparticles, and with the fixed distribution of vortices. Experimentally,[21] it is found that the longitudinal electronic thermal conductivity decreases relatively rapidly with increasing magnetic field, and then crosses over to a field independent regime at a field scale (of order a few Tesla) which depends roughly on the square of the temperature. The existence of two very different magnetic field dependences, and a crossover field scale that is very small in relation to H_{c_2}, with a small temperature scale, have proved difficult to understand. One of us has proposed, along with S. S. Mandal,[7] an explanation which is discussed in Section 5. In some systems, namely single crystal Bi-2212, Ong and coworkers found[22] that the crossover is very abrupt, and possibly discontinuous. This raises the possibility that there is some change of phase (deep within the superconducting phase in the H–T plane) as a function of field or temperature. There have been suggestions that beyond a critical (relatively small) value of the magnetic field[23] or vortex density,[24] a $\{(d_{x^2-y^2}) + id_{xy}\}$ gap function (with a fully gapped quasiparticle excitation spectrum) is stable. The magnetic field[23] or vortices[24] induce an id_{xy} order in a $d_{x^2-y^2}$ superconductor. There is no independent experimental support for this idea.

The transverse or Hall thermal conductivity[25] is hole like, and has a characteristic field and temperature dependence which is not understood. It first increases linearly with H, then seems to have a $\sqrt{H}$ dependence. It then shows a peak and starts decreasing. All this happens at low fields (of order a Tesla or so) and the peak field increases with temperature.

We thus see that the mixed state of cuprates exhibits, at low external magnetic fields, many novel equilibrium and transport phenomena. Since they occur for $H \ll H_{c_2}$ and well below T_c, the superconducting order parameter magnitude is large and homogeneous except near cores, which occupy a small fraction (H/H_{c_2}) of the volume in plane. In this limit, the $d_{x^2-y^2}$ quasiparticle couples mainly to the supercurrent distribution surrounding each vortex. This distribution is known. Further, the supercurrent varies smoothly with distance, so that it can be thought as a semiclassical field in which quasiparticles find local equilibrium. Thus the physical situation is appropriate for a semiclassical Green's function approach using an equation of motion for it.

3. CHOICES OF GAUGE:

We show here that the same Gor'kov equations for the single particle Green's function are obtained in two of the gauges mentioned in Section 2, namely the "*double valued*" symmetric gauge (Eq. (10)) and the single valued gauge where all the phase is attached to one spin species. The mean field Hamiltonian in the presence of vortices is

$$H = \int \sum_{\sigma} \psi_{\sigma}^{+}(\vec{r}) \left\{ \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 \right\} \psi_{\sigma}(\vec{r})\, d\vec{r}$$
$$+ \int d\vec{r}\, d\vec{r} \left[\tilde{\Delta}(\vec{r}, \vec{r}') \left\{ \exp\left(i \sum_{\ell} \theta \left[\left(\frac{\vec{r}+\vec{r}'}{2} \right) - \vec{R}_{\ell} \right] \right) \right\} \right.$$
$$\left. \times \psi_{\uparrow}^{+}(\vec{r})\, \psi_{\downarrow}^{+}(\vec{r}') + \text{h.c.} \right] \tag{15}$$

The Gor'kov equations are

$$\left[\nu_{\ell} - \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 \right] G(\vec{r}, \vec{r}'; \nu_{\ell})$$
$$- \int d\vec{r}\, \tilde{\Delta}(\vec{r}, \vec{r}'') \exp\left(-i\tilde{\theta}\left(\frac{\vec{r}+\vec{r}''}{2} \right) \right) F(\vec{r}'', \vec{r}; \nu_{\ell}) = \delta(\vec{r} - \vec{r}') \tag{16a}$$

$$\left[\nu_{\ell} + \frac{1}{2m}\left(\vec{p} + \frac{e\vec{A}}{c}\right)^2 \right] F(\vec{r}, \vec{r}'; \nu_{\ell})$$
$$- \int d\vec{r}''\, \tilde{\Delta}(\vec{r}, \vec{r}'') \exp\left(+i\tilde{\theta}\left(\frac{\vec{r}+\vec{r}''}{2} \right) \right) G(\vec{r}'', \vec{r}'; \nu_{\ell}) = 0 \tag{16b}$$

One way of handling the phase rotation $\tilde{\theta}$ in Eq. (16) is via the transformation

$$F(\vec{r}, \vec{r}'; \nu_{\ell}) = \exp\left\{ -i\tilde{\theta}\left(\frac{\vec{r}+\vec{r}'}{2} \right) \right\} \tilde{F}(\vec{r}, \vec{r}'; \nu_{\ell}) \tag{17a}$$

We note that if F is single valued as a function of mass the centre of coordinate $(\vec{r}+\vec{r}')$ (i.e., it does not change if $(\vec{r}+\vec{r}')/2 = \vec{R}$ goes round a vortex once) so is $\tilde{F}$. It is easy to see that if we further define

$$G(\vec{r}, \vec{r}'; \nu_{\ell}) = \exp\left\{ \frac{i}{2} (\vec{r}' - \vec{r}) \cdot \vec{\nabla}\tilde{\theta}\left(\frac{\vec{r}+\vec{r}'}{2} \right) \right\} \tilde{G}(\vec{r}, \vec{r}'; \nu_{\ell}) \tag{17b}$$

in order to eliminate an extra phase factor in, say the second term on the left side of Eq. (16a) which arises after (17a) is used, we find that the equations for $\tilde{F}$ and $\tilde{G}$ are:

$$\left\{ \nu_\ell - \frac{1}{2m}\left(\vec{p} + \frac{m\vec{v}_s}{2}\right)^2 \right\} \tilde{G}(\vec{r}, \vec{r}\,'; \nu_\ell) - \int \tilde{\Delta}(\vec{r}, \vec{r}\,'')\, \tilde{F}(\vec{r}\,'', \vec{r}\,'; \nu_\ell) = \delta(\vec{r} - \vec{r}\,') \quad (18a)$$

and

$$\left\{ \nu_\ell + \frac{1}{2m}\left(\vec{p} + \frac{m\vec{v}_s}{2}\right)^2 \right\} \tilde{F}(\vec{r}, \vec{r}\,'; \nu_\ell) - \int \tilde{\Delta}(\vec{r}, \vec{r}\,'')\, \tilde{G}(\vec{r}\,'', \vec{r}\,'; \nu_\ell) = 0 \quad (18b)$$

Equations (18a) and (18b) are exactly the equations one obtains on making the *symmetric* gauge transformation $\psi_\sigma^+(\vec{r}) \to \exp\{(i\tilde{\theta}/2)\,\mathrm{sgn}\,\sigma\}\,\tilde{\psi}_\sigma^+(\vec{r})$ where $\mathrm{sgn}\,\sigma = +$ for up spin and $-$ for down spin, directly in the mean field Hamiltonian Eq. (15). We thus see explicitly that the Green's functions which describe the effect of vortices as adding momentum $-(m\vec{v}_s/2)$ to electrons and $(-m\vec{v}_s/2)$ to holes are single valued functions of relevant coordinates and satisfy the equations of motion Eq. (18).

In the gauge Eq. (12) where all the phase is given to one electron species (say up spin) so that $\psi_\uparrow^+(\vec{r}) \to \{\exp(i\tilde{\theta})\}\,\bar{\psi}_\uparrow^+(\vec{r})$ and $\psi_\downarrow^+(\vec{r}) \to \bar{\psi}_\downarrow^+(\vec{r})$, one can transform the original Hamiltonian Eq. (15) which becomes

$$\begin{aligned} \tilde{H}^{\mathrm{GSA}} = \int d\vec{r} \Bigg[& \bar{\psi}_\uparrow^+(r)\, \frac{1}{2m}\left(\vec{p} + m\vec{v}_s + \frac{e\vec{A}}{c}\right)^2 \bar{\psi}_\uparrow \\ & + \bar{\psi}_\downarrow^+(r)\, \frac{1}{2m}\left(\vec{p} - \frac{e\vec{A}}{c}\right)^2 \bar{\psi}_\downarrow(\vec{r}) \Bigg] \\ & + \int d\vec{r}\, d\vec{r}\,' \left[\bar{\psi}_\uparrow^+(\vec{r})\, \bar{\psi}_\downarrow^+(\vec{r}\,')\, \tilde{\Delta}(\vec{r}, \vec{r}\,') \exp i\left(\frac{\vec{r}\,' - \vec{r}}{2} \cdot \vec{\nabla}\tilde{\theta}\right)\right] + \mathrm{h.c.} \end{aligned} \quad (19)$$

The kinetic energy term has obvious asymmetries, as has the pair potential term showing up there an additional, relative coordinate $(\vec{r} - \vec{r}\,')$ dependent phase which is not negligible. The Gor'kov equations with $\tilde{H}^{\mathrm{GSA}}$ carry this phase with them. We see again that a permissible transformation

$$F^{\mathrm{GSA}}(\vec{r}, \vec{r}\,'; \nu_\ell) \to \exp[i(\vec{r} - \vec{r}\,') \cdot \vec{\nabla}\tilde{\theta}/2]\, \bar{F}(\vec{r}, \vec{r}\,'; \nu_\ell) \quad (20a)$$

and

$$G^{\mathrm{GSA}}(\vec{r}, \vec{r}\,'; \nu_\ell) \to \exp[i(\vec{r} - \vec{r}\,') \cdot \vec{\nabla}\tilde{\theta}/2]\, \bar{G}(\vec{r}, \vec{r}\,'; \nu_\ell) \quad (20b)$$

leads to the Gor'kov equations for $\bar{F}$ and $\bar{G}$ which are identical with those for $\tilde{F}$ and $\tilde{G}$ (Eq. (18)).

Gor'kov like equations in the presence of external potentials can also be derived similarly, e.g., for the "correlation" functions $g^{<}(1,2) = i\langle \psi^{+}(2)\,\psi(1)\rangle$, and they too have the same superfluid momentum term in the presence of vortices. As is well known[5, 6], such equations can be used to define the quasiparticle spectral functions, and to obtain a semiclassical transport equation. In all of them, the effect of vortices is to add a momentum $(m\vec{v}_s/2)$ as shown above.

The physical reason for the singlevaluedness of $\tilde{G}$ and $\tilde{F}$ is obviously that they are *quadratic* in the field, so that even if the transformed fields (with half flux quanta attached) are double valued, $\tilde{G}$ and $\tilde{F}$ are not. Further, the functions $\tilde{G}(\vec{r}, \vec{r}\,'; v_\ell)$ and $\tilde{F}(\vec{r}, \vec{r}\,'; v_\ell)$ are short ranged as a function of relative coordinate $(\vec{r} - \vec{r}\,')$; the range is of order the lattice spacing a. Thus $\vec{r}$ and $\vec{r}\,'$ are very close to each other. The effect of vortices is to cause a phase rotation in cm coordinates $\{\vec{r} + \vec{r}\,'/2\}$. The smallest circuit of interest here, in the cm coordinates, has a radius ξ (coherence length) $\simeq 5$ to $10a$. Gauge covariance of Gor'kov equations in a magnetic field has been discussed earlier by Ryan and Rajagopal.[26]

In relating the various Green's functions above, we have effectively expanded quantities as a power series in the ratio (a/ξ) and have kept the first term. This is a semiclassical approximation. The results obtained from this approximation may not be correct, e.g., at the lowest energies for the density of quasiparticle states.[14, 15]

4. SINGLE PARTICLE SPECTRUM

It is well known[5, 25] that the equations of motion of one particle correlation functions, e.g.,

$$G_{11}^{<}(1,2) = i\langle \psi_{\uparrow}^{+}(x_2 \psi_{\uparrow}(x_1))\rangle \tag{21}$$

can be used to find the single particle spectral density and the transport equation satisfied by the one particle distribution function. The idea is to express the equations of motion (one obtained by taking the time derivative with respect to t_1, and another by taking the time derivative with respect to t_2) in terms of difference or relative coordinate variables $(x_1 - x_2) = (\vec{r}, t)$, and the centre of mass variables $(x_1 + x_2/2) = (\vec{R}, T)$. Physical quantities vary rapidly as a function of $\vec{r}$ and t (on a scale $|\vec{r}| \sim a \sim k_F^{-1}$, and $t \sim \varepsilon_F^{-1}$) but slowly as a function of $\vec{R}$, T (the natural scale here is $|R| > \xi$, and $T > \Delta_o^{-1}$). One then works with Fourier transforms in $(\vec{r}, t)$ variables, $(\vec{k}, \omega)$ say. These correspond to the quasiparticle wavevector and frequency respectively. Because of the slow variation in $(\vec{R}, T)$, one makes a gradient expansion where necessary, and keeps the leading terms in $\vec{r} \cdot \vec{\nabla}_{\vec{R}} \sim (a/R^*) \sim (a/\xi)$. Since this formalism is well known both in its Green's function[5, 27] and Keldysh[6] versions and has indeed been used to obtain quasiparticle density of states and the transport equation for an s-wave superconductor[6] we mention here a typical equation which enables us to extract results of interest.

The equation of motion for the (2×2) matrix Green's function $\tilde{G}^<(1, 2)$ can be written[27] in the Nambu notation as

$$\left[i\hbar \frac{\partial}{\partial t_1} I - \left\{ (2m)^{-1} \left(\vec{p} + \frac{m\vec{v}_s(x_1)}{2} \tau_3 \right)^2 + V(x_1) - \mu \right\} \tau_3 \right] \tilde{G}^<(x_1, x_2)$$

$$- \int dx' \, \tilde{\Delta}(x_1, x') \, \tau_1 \tilde{G}^<(x', x_2) = 0 \tag{22}$$

We have assumed no short range quasiparticle scattering, and real $\tilde{\Delta}(x_1, x')$. It is also realistic to assume that for low vortex density, $\tilde{\Delta}(x, x')$ is that appropriate for a homogeneous superconductor, i.e., $\tilde{\Delta}(x, x') \simeq \tilde{\Delta}(\vec{x} - \vec{x}\,') = \sum_k \Delta_k \exp i\vec{k} \cdot (\vec{x} - \vec{x}\,')$. Using the difference and sum variables as mentioned in the previous paragraph, namely using $(\partial/\partial t_{1,2}) = \pm(\partial/\partial t) + (1/2)(\partial/\partial T)$; *and* $\vec{p}_{1,2} = (\vec{P}/2) \pm \vec{p}$; $\vec{r}_{1,2} = \vec{R} \pm (\vec{\rho}/2)$, we reexpress Eq. (22) in terms of these variables. An equation similar to Eq. (22) is obtained on differentiating $\tilde{G}^<(x_1, x_2)$ with respect to t_2. On subtracting this equation from Eq. (22), we find

$$\left[i\hbar \frac{\partial}{\partial t} I - \{ (\hbar^2 k^2/2m) - \mu + V(\vec{R}) + (mv_s(\vec{R})^2/2) \} \, \tau_3 \right.$$

$$\left. - \hbar\vec{k} \cdot \vec{v}_s(\vec{R}) \, I - \Delta_{\vec{k}} \tau_1 \right] \tilde{G}(\vec{k}, \vec{R}, t) = 0 \tag{23}$$

where derivatives of higher order in $\vec{\nabla} R$ are be neglected. Suppose the external potential $V(\vec{R}) = 0$. Since $\vec{v}_s$ is a small quantity, the quadratic term (or at least its spatially fluctuating component) can be neglected. One thus concludes on diagonalizing Eq. (23) that (not surprisingly) there continue to be well defined quasiparticles with frequency ω such that

$$\hbar\omega = \pm \sqrt{(\varepsilon_k - \mu)^2 + \Delta_k^2} + \hbar\vec{k} \cdot \vec{v}_s(\vec{R}) = \pm E_k^o + \hbar\vec{k} \cdot \vec{v}_s(\vec{R}) \tag{24}$$

This is the Volovik result. We have placed it explicitly in the context of a semiclassical Green's function approach. The quasiparticle frequency depends on both $\vec{k}$ and $\vec{R}$. Depending on where the quasiparticle is (i.e., $\vec{R}$), the wavevector $\vec{k}$ changes for a fixed energy, i.e., the quasiparticle is scattered from one momentum state to another as it moves through the vortex medium.

We now consider evaluating the density of states of such quasiparticles, for a general distribution of vortices. We are interested in

$$\rho(\nu) = \sum_{\pm, \vec{k}} \langle \delta[\nu \pm E^o_k + \hbar\vec{k} \cdot \vec{v}_s(\vec{R})] \rangle \tag{25}$$

In Eq. (25) the angular bracket average means averaging over vortex positions. One way of doing this is to write the δ function as a Fourier integral, so that

$$\rho(\nu) = \frac{1}{2\pi} \sum_{\pm, \vec{k}} \left\langle \int_{-\infty}^{+\infty} dt\, e^{i[\nu \pm E^o_k + \hbar\vec{k} \cdot \vec{v}_s(\vec{R})]\, t} \right\rangle \tag{26}$$

The statistical average of the exponential, i.e.,

$$\langle \exp[i\hbar\vec{k} \cdot \vec{v}_s(\vec{R})\, t] \rangle$$

can be found using cumulants or irreducible vortex-vortex correlation functions. Considering only the two vortex irreducible correlation function to be nonzero, we find

$$\rho(\nu) = \frac{1}{\sqrt{\pi}} \sum_{k, \pm} \frac{1}{\bar{\varepsilon}_k} e^{-(\nu \pm E^o_k)^2/\bar{\varepsilon}^2_k} \tag{27a}$$

where

$$\bar{\varepsilon}^2_k = \frac{n_v}{2m^2 A} \sum_{\vec{q}} \left[\frac{\hbar\vec{k} \cdot (\hat{e}_z \times \vec{q})}{q^2 + \lambda^{-2}} \right]^2 S_q \tag{27b}$$

$\bar{\varepsilon}_k$ is the characteristic "broadening" of the spectral function which depends on the vortex density n_v, and how the vortices are arranged, S_q being the vortex-vortex structure factor

$$S_q = (1/N) \left\langle \sum_{\ell} \exp(i\vec{q} \cdot \vec{R}_\ell) \sum_j \exp(-i\vec{q} \cdot \vec{R}_j) \right\rangle \tag{28}$$

The energy scale $\bar{\varepsilon}_{k_F}$ is of order $\bar{\varepsilon}_{k_F} \simeq \sqrt{\varepsilon_F(\hbar^2 n_v/m)}$. For a field of $1T$, and with $\varepsilon_F \simeq 0.5eV$, $\bar{\varepsilon}_{k_F} \sim 40K$, a fairly large value approximately equal to $(T_c/2)$. At zero energy, the density of states per unit area A

$$\rho(0) = (\pi/v_F v_\Delta)\, \bar{\varepsilon}_{k_F} \tag{29}$$

as if (see Eq. (4a)) the quasiparticle is at an effective energy $\bar{\varepsilon}_{k_F}$.

The result Eq. (27) obviously implies that the free energy of the mixed phase superconductor depends on how the vortices are arranged, not only via the intervortex potential, but also through how vortex order affects the quasiparticle spectrum and therefore the electronic free energy. A calculation[4] shows that the crystalline solid state has lower quasiparticle density of states, and is therefore more stable at low temperature, than the vortex fluid. The transition temperature $T_{fs} \sim H^2$. It is possible that this is the transition observed as a kink in electronic thermal conductivity as a function of field in some systems.[21] It also appears that the paramagnetic reduction in superfluid stiffness due to quasiparticle excitation depends on vortex matter structure. A marked decrease in superfluid stiffness (via penetration depth measurements) has been seen on crossing the vortex solid-fluid boundary, well below $T_c(H=0)$. The result Eq. (27) is directly useful for exploring such questions.

5. TRANSPORT

A transport equation for quasiparticles coupled to the slowly varying supercurrent $\vec{v}_s(\vec{R})$ is obtained by adding Eq. (23) and a similar one which describes the time derivative of $\widetilde{G}(x_1, x_2)$ with respect to t_2. We find that

$$\left[\frac{\partial}{\partial T} + \vec{v}_k \cdot \vec{\nabla}_R + v_{k\propto}\{\partial(mv_s^{\propto}(R))/\partial R_\beta\}(\partial/\partial k_\beta)\right] f(\omega, \vec{k}, \vec{R}, T)$$
$$= (\partial f/\partial T)_{\text{coll}} \tag{30}$$

This is the same as the equation derived by Aronov *et al.*[6] using the Keldysh technique, for moving vortices, and an s-wave superconductor. In the presence of a temperature gradient $\vec{\nabla}T$, the second term in the left side of Eq. (30) is linear in it.

Interesting quasiparticle relaxation effects arise, even in the absence of quasiparticle collisions from other scatterers ($\dot{f}_{\text{coll}} = 0$), due to the presence of the third, Lorentz magnetic force like term involving the spatial derivative of the superfluid velocity. This term can be cast in the form

$$\left[\left(\frac{e}{c}\right)\vec{v}_k \times \vec{H}_{\text{eff}}\right]_\beta \cdot (\partial f/\partial k_\beta) \tag{31a}$$

where

$$\vec{H}_{\text{eff}} = \vec{\nabla} \times \vec{A}_{\text{eff}} = \vec{\nabla} \times (-(mc\vec{v}_s/2e)) \tag{31b}$$

It is clear that $\bar{H}_{\text{eff}} = 0$ because of London screening. However, the spatially fluctuating parts of it cause quasiparticle scattering. For the s-wave superconductor, the longitudinal transport cross-section, and the transverse or skew scattering cross-section have been calculated for one vortex by Aronov *et al.*[6] in a classical approach, i.e., by using classical trajectories satisfying the secular Hamiltonian condition

$$\hbar\omega = E^o_{\vec{k}} + \hbar\vec{k}\cdot\vec{v}_s(\vec{R}) = \text{constant} \tag{32}$$

This classical problem is different here because $E^o_{\vec{k}}$ is anisotropic. Further, the independent vortex or vortex gas model is a poor approximation to the dense vortex system, and leads to qualitative errors[7] for transport behavior at common vortex densities.

We have investigated[7] the quasiparticle transport equation with the scattering from vortices calculated in the Born approximation. This is known to give the exact differential longitudinal transport scattering cross section for one vortex (in s-wave superconductors) because of the long range of the interaction, but no transverse or skew scattering. The observed $K_{xx}(H, T)$ behavior[21] is explained qualitatively and quantitatively. At low fields, the relaxation rate due to scattering from vortices is proportional to their number, i.e., to H. This causes the observed initial $(1/H)$ like decrease of $K_{xx}(H, T)$ with H. At low T or large H, the maximum available momentum transfer for elastic scattering is small, so that vortices do not scatter independently, but through long wavelength compressional fluctuations of the dense collection of vortices. These are highly suppressed, with a size inversely dependent on vortex density. This leads to a field independent K_{xx}. The crossover field is also correctly given.

The Hall thermal conductivity arises from the spatially fluctuating transverse force in Eq. (31), convolved with fluctuating terms in the distribution function due eg. to position dependent quasi-particle energy, to finally give a uniform contribution. The sign and field dependence observed[25] can be obtained this way.[7]

Many transport phenomena in the mixed phase of cuprates, e.g., the electrical or flux flow resistivity (Hall and longitudinal) are likely to be connected with the interaction between quasiparticles *outside* the vortex core and the supercurrents associated with vortices (moving, in presence of an electric field). This is in contrast to conventional superconductors, where it is believed that dissipation is due to the normal state quasiparticles *inside* the vortex core. In cuprates, because the gap has nodes, there are gapless or low energy quasiparticle excitations outside the vortex core; these interact with the slowly varying superflow associated with the vortices (moving or fixed). Further, since the vortex core is small, there are very few states (one?) inside it in contrast to conventional superconductors with large ξ which support a large number, or a nearly continuous spectrum of core states (piece of normal metal). Thus the nature of vortex dynamics, flux flow dissipation, Hall resistivity etc. is likely to be qualitatively different from that in s-wave superconductors, as appears to be the case experimentally (see, e.g., ref. 20). We believe that the semiclassical approach used here will be the natural one for such problems.

There is of course a large class of phenomena involving quasiparticles in the superconducting state where a semiclassical approach is inadequate. We have already mentioned an example, the bound quasiparticle state[18, 19] in a vortex core. The density of states for low quasiparticle energies in the presence of vortices and disorder, and the question of their localization are some other examples. The high magnetic field behavior of cuprate superconductors and the transition to the normal state brings us to the strange cuprate metal, which might be related to a Luttinger liquid.

We join our colleagues in remembering J. M. Luttinger. One of us (T.V.R.) was his Ph.D. student. Professor Luttinger's clarity and insight, his focus on fundamental questions, the elegance and power of his approach to problems in physics, and the timeless quality of his papers, have all left a vivid and lasting impression. His contributions which have given shape to quantum many body theory, continue to influence new departures in many-body physics. Luttinger was also a caring advisor and friend. This paper is dedicated to the memory of a great physicist, an exceptional teacher and a wonderful friend.

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